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FIRST YEAR NOTES

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Date: / /

Laplace transform

If $f(t)$ is a function defined for all $t>0$ then the Laplace transform of $f(t)$ is defined by

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

The given function $f(t)$ is called the inverse Laplace transform of $F(s)$ and is denoted by

$$L^{-1}\{F(s)\} = f(t)$$

Existence and Uniqueness Theorem

If $f(t)$ is piecewise continuous and $|f(t)| \leq Me^{kt}$ for all $t>0$, then Laplace transform $L\{f(t)\}$ excist for all $s>k$ and is unique.

Linearity Property

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$$

1) $L(1) = \int_0^\infty e^{-st} \times 1 \, dt$

$= \int_0^\infty e^{-st} dt$

$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$

$= \frac{0-1}{-s} = \frac{1}{s}, \quad s>0$

2) $L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt$

$= \int_0^\infty e^{-(s-a)t} dt$

$= \left[\frac{e^{-(s-a)}}{-(s-a)} \right]_0^\infty$

$= \frac{0-1}{-(s-a)} = \frac{1}{s-a}, \quad s>a$

3) $L(t) = \int_0^\infty e^{-st} \times t \, dt$

$= \left[t \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 1 \times \frac{e^{-st}}{-s} dt$

$= 0 + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^\infty$

$= \frac{0-1}{-s^2} = \frac{1}{s^2}, \quad s>0$

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Date: / / /

* Laplace transforms of some basic functions.

1. Find the Laplace transform of $f(t) = 1$ for $t \geq 0$

Soln

$$L\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, \; s > 0$$

2. Find the Laplace transform of $f(t) = e^{at}$, $t \geq 0$

$$L\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

3. Evaluate $L(t)$

$$L\{t\} = \frac{1}{s^2}, \quad s > 0$$

4. Evaluate $L\{t^n\}$, n is a non-negative integer

$$L\{t^n\} = \int_0^\infty t^n e^{-st}\, dt$$

$$= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\, dt$$

$$= 0 + \frac{n}{s} L(t^{n-1})$$

$$= \frac{n}{s} \frac{n-1}{s} L(t^{n-2})$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{2}{s} L(t)$$

$$= \frac{n}{s} \frac{n-1}{s} \cdots \frac{2}{s} \cdot \frac{1}{s^2}$$

$$= \frac{n!}{s^{n+1}}, \quad s>0$$

5. Evaluate $L\{\sin at\}$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$= 0 + \frac{a}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2}, \quad s>0$$

6. $L\{\cos at\} = \dfrac{s}{s^2+a^2}, \quad s>0$

Date: / /

7. Evaluate $L\{\sinh at\}$

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) - (s-a)}{s^2 - a^2}\right]$$

$$= \frac{a}{s^2 - a^2}, \quad s > a$$

8. $L(\cosh at) = \dfrac{s}{s^2 - a^2}, \quad s > a$

$$L(\cosh at) = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2} L(e^{at}) + \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} + \frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a) + (s-a)}{s^2 - a^2}\right]$$

$$= \frac{s}{s^2 + a^2}, \quad s > a$$

Date : / / /

Qn Evaluate $L^{-1}\left(\dfrac{5s+1}{s^2-25}\right)$

Soln :

$$\frac{5s+1}{(s-5)(s+5)} = \frac{A}{s-5} + \frac{B}{s+5}$$

$$5s+1 = A(s+5) + B(s-5)$$

$$A+B = 5$$

$$5A-5B = 1 \qquad \therefore \; A = 13/5$$

$$B = 12/5$$

$$\therefore \; L^{-1}\left(\frac{5s+1}{s^2-25}\right) = L^{-1}\left[\frac{13/5}{s-5} + \frac{12/5}{s+5}\right]$$

$$= \frac{13}{5} L^{-1}\left(\frac{1}{s-5}\right) + \frac{12}{5} L^{-1}\left(\frac{1}{s+5}\right)$$

$$= \frac{13}{5} e^{5t} + \frac{12}{5} e^{-5t}$$

Qn Evaluate $L^{-1}\left(\dfrac{s^2+2s+5}{s^3}\right)$

$$L^{-1}\left(\frac{s^2+2s+5}{s^3}\right) = L^{-1}\left(\frac{s^2}{s^3} + \frac{2s}{s^3} + \frac{5}{s^3}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) + 2L^{-1}\left(\frac{1}{s^2}\right)$$

$$+ 5L^{-1}\left(\frac{1}{s^3}\right)$$

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$$= 1 + 2\frac{t}{1!} + 5\frac{t^2}{2!}$$

$$= 1 + 2t + \frac{5}{2}t^2$$

Qn. Evaluate $L^{-1}\left[\frac{3s+2}{s^2+9}\right]$

Soln

$$L^{-1}\left[\frac{3s+2}{s^2+9}\right] = L^{-1}\left(\frac{3s}{s^2+9}\right) + L^{-1}\left(\frac{2}{s^2+9}\right)$$

$$= 3L^{-1}\left(\frac{s}{s^2+9}\right) + 2L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= 3\cos 3t + \frac{2}{3}\sin 3t$$

Date : / / /

First Shifting Theorem

If $F(s)$ is the Laplace transform of $f(t)$, then $L\{e^{at} f(t)\} = F(s-a)$

This is equivalent to

$$L^{-1}\{F(s-a)\} = e^{at} f(t)$$

Qn Prove the first shifting theorem

GT $L\{f(t)\} = F(s)$

$$\therefore L\{e^{at} f(t)\} = \int_0^\infty e^{at} f(t) e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= F(s-a)$$

Qn Evaluate $L(e^{2t} \cos 3t)$

Soln:

$$L(\cos 3t) = \frac{s}{s^2+9}$$

$$\therefore L(e^{2t} \cos 3t) = \frac{s-2}{(s-2)^2+9} \text{ by FST}$$

Date: /

Qn Evaluate $L(e^{2t} \sin 3t)$

Soln:

$$L(\sin 3t) = \frac{3}{s^2+9}$$

$$L(e^{2t} \sin 3t) = \frac{3}{(s-2)^2+9}$$

Qn Evaluate $L(t^2 e^{-3t})$

Soln

$$L(t^2) = \frac{2!}{s^3}$$

$$\therefore L(e^{-3t} t^2) = \frac{2}{(s+3)^3}$$

Qn. Find $L^{-1}\left(\frac{2s-1}{s^2-6s+13}\right)$

Soln:

$$\frac{2s-1}{s^2-6s+13} = \frac{2(s-3)+5}{(s-3)^2+2^2}$$

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Date: / / /

$$\therefore L^{-1}\left(\frac{2s-1}{s^2-6s+13}\right) = 2L^{-1}\left(\frac{s-3}{(s-3)^2+2^2}\right) + 5\,L^{-1}\left(\frac{1}{(s-3)^2+2^2}\right)$$

$$= 2e^{3t}\cos 2t + \frac{5}{2}e^{3t}\sin 2t$$

Date: / /

Qn Find $L^{-1}\left(\frac{3s-137}{s^2+2s+401}\right)$

Solution

$$\frac{3s-137}{s^2+2s+401} = \frac{(3s+3)-140}{(s+1)^2+20^2}$$

$$= \frac{3(s+1)-140}{(s+1)^2+20^2}$$

$$\therefore L^{-1}\left(\frac{3s-137}{s^2+2s+401}\right) = L^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+20^2}\right)$$

$$= 3L^{-1}\left(\frac{\cancel{3}(s+1)}{(s+1)^2+20^2}\right) - 140L^{-1}\left(\frac{1}{(s+1)^2+20^2}\right)$$

$$= 3e^{-t}\cos 20t - 140 \times e^{-t}\frac{\sin 20t}{20}$$

$$= e^{-t}\left[3\cos 20t - 7\sin 20t\right]$$

Date: / / /

* Change of Scale Property

If $L[f(t)] = F(s)$ and c is any positive constant, then

$$L[f(ct)] = \frac{1}{c} F\left(\frac{s}{c}\right)$$

Qn. Using change of scale property evaluate $L(e^{-2t}\sin^2(2t))$, given that $L(e^{-t}\sin^2(t)) = \frac{2}{(s+1)(s^2+2s+5)}$

<u>Soln.</u>

Let $L(e^{-t}\sin^2(t)) = \frac{2}{(s+1)(s^2+2s+5)} = F(s)$

Then by change of scale property.

$$L(e^{-2t}\sin^2(2t)) = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2} \times \frac{2}{\left(\frac{s}{2}+1\right)\left(\left(\frac{s}{2}\right)^2 + 2\frac{s}{2} + 5\right)}$$

$$= \frac{8}{(s+2)(s^2+4s+20)}$$

* Laplace transform of Derivatives

$$L[f'(t)] = s\,L[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$$

$$\vdots$$

$$L[f^{(n)}(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots$$
$$\cdots - f^{(n-1)}(0)$$

Qn Using Laplace transform of derivatives, evaluat $L[f(t)]$ where $f(t) = t \sin at$.

Soln

$$f(t) = t \sin at$$
$$f'(t) = \sin at + at \cos at$$
$$f''(t) = a\cos at + a\cos at - a^2 t \sin at$$
$$= 2a\cos at - a^2 t \sin at$$

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Date: / /

Also $f(0) = f'(0) = 0$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[-a^2 t \sin at + 2a \cos at] = s^2 L[f(t)]$$

$$-a^2 L(f(t)) + 2a\, L(\cos at) = s^2 L(f(t))$$

$$\therefore (s^2+a^2) L[f(t)] = \frac{2as}{s^2+a^2}$$

$$L[f(t)] = \frac{2as}{(s^2+a^2)^2}$$

Qn. Solve the IVP $y'' - y = t$, $y(0)=1$, $y'(0)=1$

Soln.

GT $\quad y'' - y = t$

$$\therefore L(y'' - y) = L(t)$$

$$L(y'') - L(y) = L(t)$$

$$s^2 L(y) - sy(0) - y'(0) - L(y) = \frac{1}{s^2}$$

Date : / / /

$$(s^2-1)L(y) - s - 1 = \frac{1}{s^2}$$

$$(s^2-1)L(y) = \frac{1}{s^2} + s + 1$$

$$L(y) = \frac{1}{s^2(s^2-1)} + \frac{s+1}{(s^2-1)}$$

$$= \frac{-1}{s^2} + \frac{1}{(s^2-1)} + \frac{1}{(s-1)}$$

$$\therefore y(t) = -L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{1}{s^2-1}\right) + L^{-1}\left(\frac{1}{s-1}\right)$$

$$= -t + \sinh t + e^t$$

* Shifted IVP

If IVP is of the form $y''(t) + ay'(t) + by(t) = R(t)$, $y(t_0) = k_0$ and $y'(t_0) = k_1$, where $t_0 > 0$

Then we define new variables $\bar{y}$ and $\bar{t} = t - t_0$ such that $\bar{y}(\bar{t}) = y(t)$

$\therefore y'' + ay' + by = G(\bar{t})$

When $t = t_0$, $\bar{t} = 0$

$$\therefore y(t_0) = \bar{y}(0)$$

$$\& \; y'(t_0) = \bar{y}'(0)$$

Date: / /

Qn. Solve the IVP

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \ y'\left(\frac{\pi}{4}\right) = 2-$$

Soln:

GT $y'' + y = 2t$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, $y'\left(\frac{\pi}{4}\right) = 2-$

Here $t_0 = \pi/4$

Let $\bar{t} = t - \frac{\pi}{4}$ and $\bar{y}(\bar{t}) = y(t)$

$\therefore$ The given problem can be written as

$$\bar{y}'' + \bar{y} = 2\left(\bar{t} + \frac{\pi}{4}\right),$$

$$\bar{y}(0) = y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\bar{y}'(0) = y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

Applying LT

$$L(\bar{y}'') + L(\bar{y}) = 2L(\bar{t}) + \frac{2\pi L(1)}{4}$$

$$s^2 L(\bar{y}) - s\,\bar{y}(0) - \bar{y}'(0) + L(\bar{y}) = 2 \times \frac{1}{s^2} + \frac{\pi}{2} \times$$

Date: / / /

$$(s^2+1)\,L(\bar{y}) - s\frac{\pi}{2} - (2-\sqrt{2}) = \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

$$(s^2+1)\,L(\bar{y}) = \frac{2}{s^2} + \frac{\pi}{2}\left(s+\frac{1}{s}\right) + (2-\sqrt{2})$$

$$= \frac{2}{s^2} + \frac{\pi}{2}\left(\frac{s^2+1}{s}\right) + (2-\sqrt{2})$$

$$L(\bar{y}) = \frac{2}{s^2(s^2+1)} + \frac{\pi}{2}\frac{1}{s} + \frac{2-\sqrt{2}}{s^2+1}$$

$$\therefore \bar{y}(\bar{t}) = 2L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] + \frac{\pi}{2}L^{-1}\left(\frac{1}{s}\right) + (2-\sqrt{2})\,L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 2L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] + \frac{\pi}{2}L^{-1}\left(\frac{1}{s}\right) + (2-\sqrt{2})\,L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 2L^{-1}\left(\frac{1}{s^2}\right) + \frac{\pi}{2}L^{-1}\left(\frac{1}{s}\right) \cancel{+} -\sqrt{2}\,L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 2\bar{t} + \frac{\pi}{2} - \sqrt{2}\sin\bar{t}$$

$$\sin\bar{t} = \sin\left(t-\frac{\pi}{4}\right)$$

$$= \sin t\cos\frac{\pi}{4} - \cos t\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}(\sin t - \cos t)$$

$$= 2\left(t-\frac{\pi}{4}\right) + \frac{\pi}{2} - \frac{\sqrt{2}}{\sqrt{2}}\left[\sin t - \cos t\right]$$

$$= 2t - \sin t + \cos t$$

* Laplace transform of integral

If $F(s)$ is the Laplace transform of $f(t)$ then

$$L\left[\int_0^t f(x)\,dx\right] = \frac{F(s)}{s}$$

This is equivalent to

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(x)\,dx$$

Qn. Evaluate $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Soln

We have $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

$$\therefore L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \int_0^t \frac{\sin at}{a}\,dt$$

$$= \frac{-1}{a}\left[\frac{\cos at}{a}\right]_0^t$$

$$= \frac{-1}{a^2}\left[\cos at - 1\right]$$

$$= \frac{1-\cos at}{a^2}$$

Date: / / /

Qn. Using Laplace transform of derivatives, evaluate $L(t\cos at)$

Qn. Find $L^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right)$

Qn. Solve $y''+9y = 10e^{-t}$, $y(0)=0$, $y'(0)=0$

Qn Solve $y''-2y'-3y=0$, $y(4)=-3$, $y'(4)=-17$

Date: /

* ## Heaviside Function

The Heaviside function or unit step fu[illegible] is defined by

$$u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t>a \end{cases}$$

where $a \geq 0$

Qn. Find $L[u(t-a)]$

$$L(u(t-a)) = \int_0^\infty u(t-a)\, e^{-st}\, dt$$

$$= \int_0^a u(t-a)\, e^{-st}\, dt + \int_a^\infty u(t-a) e^{-st} dt$$

$$= 0 + \int_a^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

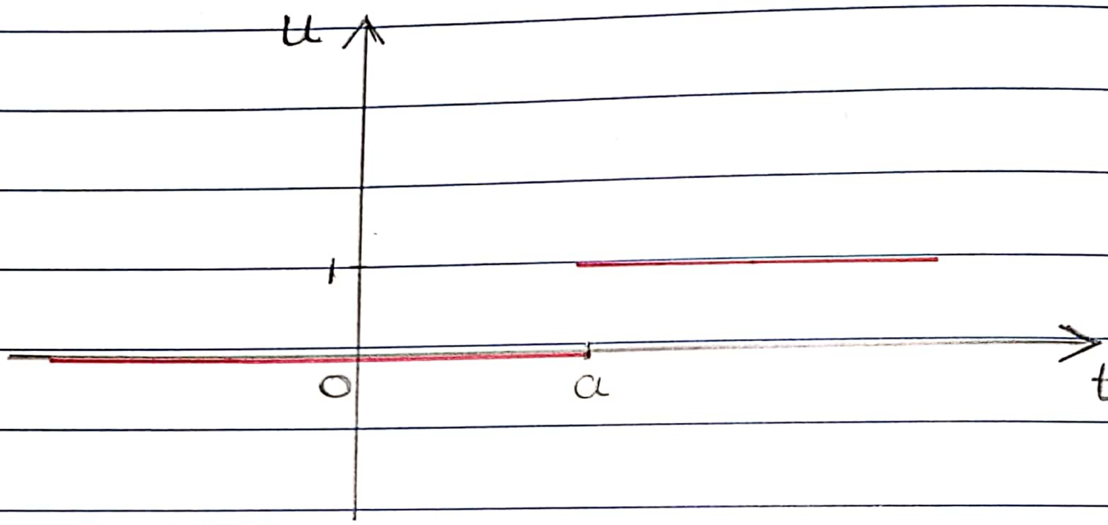
$$= \frac{0 - e^{-as}}{-s}$$

$$= \frac{e^{-as}}{s}, \quad s>0$$

Date : / /

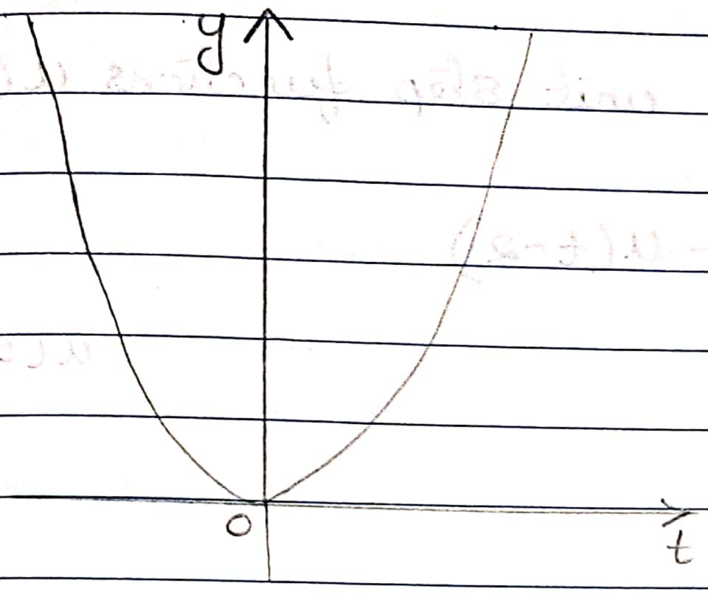
Unit Step Function

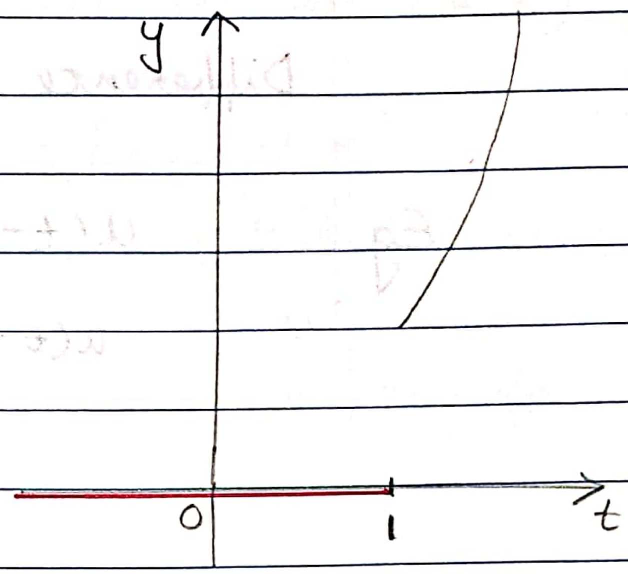
$$u(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t>a \end{cases}, \quad a \geq 0$$



Date : / / /

Consider $y = t^2$ $\qquad$ $y = t^2 u(t-1)$





When $t < 1$, $u(t-1) = 0$ $\quad \therefore \; y = t^2 u(t-1) = 0$

$t > 1$, $u(t-1) = 1$ $\quad y = t^2 u(t-1) = t^2$

In general

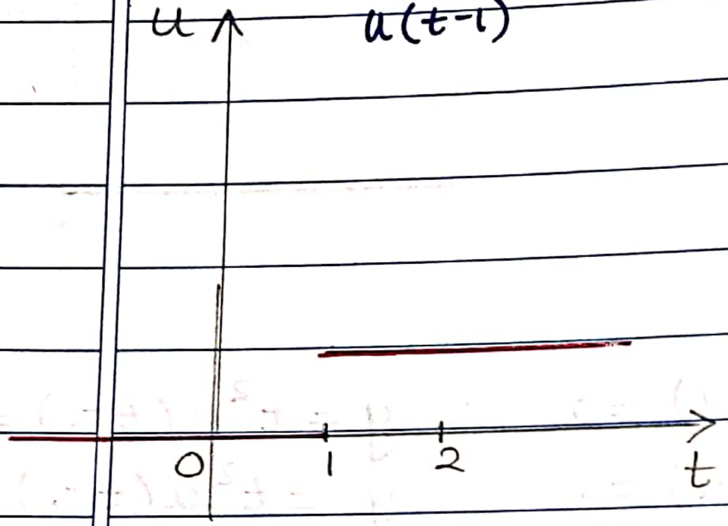
$$f(t)\,u(t-a) = \begin{cases} 0 & , t < a \\ f(t) & , t > a \end{cases}$$

Property of unit step function.

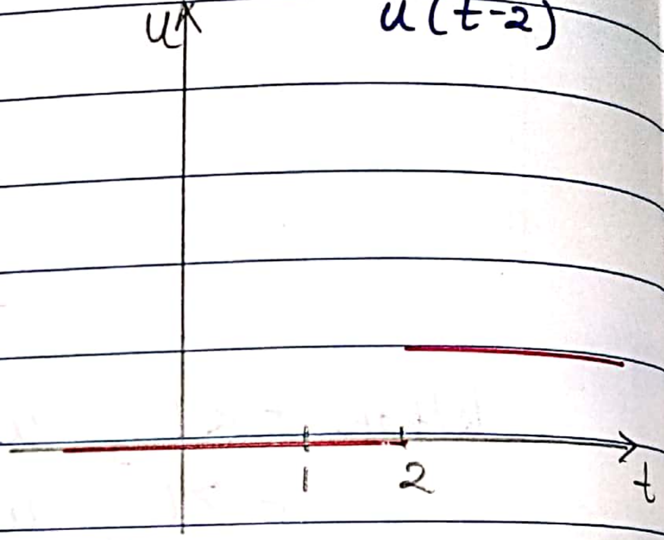
Difference of unit step functions $u(t-a)-u(t-b)$

Eg $u(t-1) - u(t-2)$

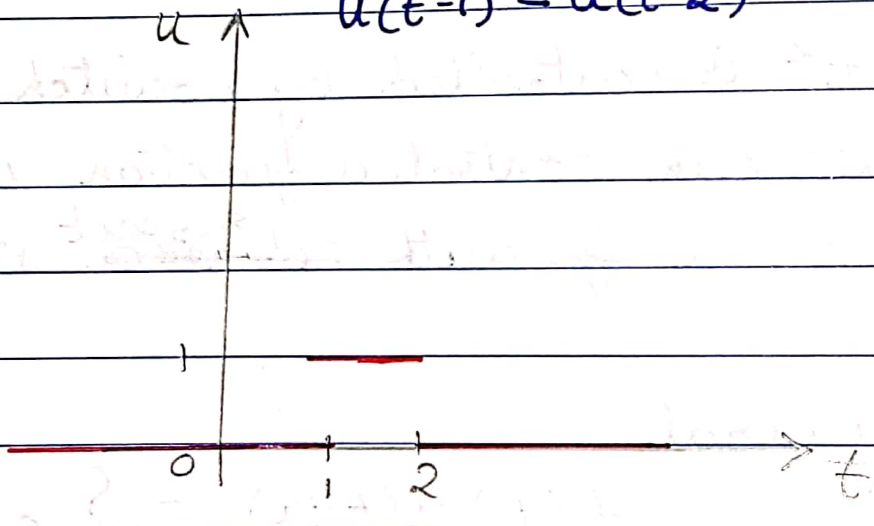
$u(t-1)$



$u(t-2)$



$u(t-1) - u(t-2)$

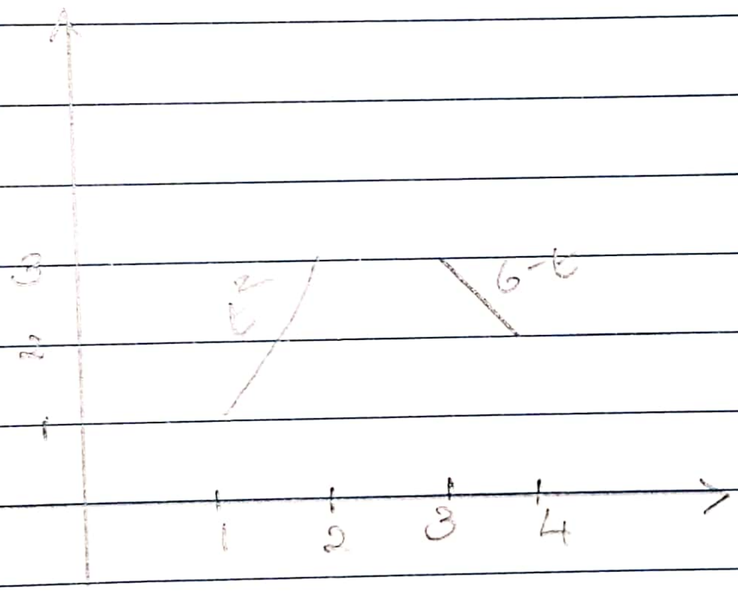


Date: / / /

Suppose we have to turn on t^2 in the interval $(1,2)$ and $6-t$ in the interval $(3,4)$

$$t^2\,[\text{switch } 1] + (6-t)[\text{switch } 2]$$
$$t^2\left[u(t-1)-u(t-2)\right] + (6-t)\left[u(t-3)-u(t-4)\right]$$

* Thus we can turn off and turn on any fn in any interval.



Date: / / /

✱ Second Shifting theorem

If $F(s)$ is the Laplace transform of $f(t)$ then

$$L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)] = e^{-as} F(s)$$

which is equivalent to

$$L^{-1}[e^{-as} F(s)] = f(t-a)\,u(t-a)$$

Proof

$$L\left[f(t-a)\,u(t-a)\right] = \int_0^\infty f(t-a)\,u(t-a)\,e^{-st}\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Let $v = t-a$

$dv = dt$

$$= \int_0^\infty e^{-s(v+a)} f(v)\,dv$$

$$= e^{-sa}\int_0^\infty e^{-sv} f(v)\,dv$$

$$= e^{-sa} F(s)$$

Qn. Find $L[3u(t-2)\cos(t-2)]$

Soln

We know that $L(\cos t) = \frac{s}{s^2+1} = F(s)$

By 2^{nd} ST we have

$$L(3u(t-2)\cos(t-2)) = 3L(\cos(t-2)u(t-2))$$

$$= 3e^{-2s}F(s)$$

$$= 3e^{-2s}\frac{s}{s^2+1}$$

Qn. Write the following function using unit step function, hence evaluate its Laplace transf

$$f(t) = \begin{cases} e^t & \text{for } 0<t<2 \\ 0 & \text{for } t>2 \end{cases}$$

Soln.

$$f(t) = e^t \left[u(t-0) - u(t-2)\right]$$

$$= e^t \left[u(t) - u(t-2)\right]$$

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Date: / / /

$$= e^t\left[1 - u(t-2)\right]$$

$$= e^t - e^{t-2+2}u(t-2)$$

$$= e^t - e^2 e^{(t-2)}u(t-2)$$

$$\therefore L[f(t)] = L(e^t) - e^2 L\left[e^{(t-2)}u(t-2)\right]$$

$$= \frac{1}{s-1} - e^2 e^{-2s} L(e^t) \quad (\text{By } 2^{nd} \text{ ST})$$

$$= \frac{1}{s-1} - e^{2(1-s)} \times \frac{1}{s-1}$$

$$= \frac{1-e^{2(1-s)}}{s-1}$$

Qn Find $L^{-1}\left(\frac{e^{-2s}\pi}{s^2+\pi^2} + 5\frac{e^{-s}}{s^2+\pi^2}\right)$

Soln:

$$= L^{-1}\left(\frac{e^{-2s}\pi}{s^2+\pi^2}\right) + 5L^{-1}\left(\frac{e^{-s}}{s^2+\pi^2}\right)$$

$$= L^{-1}\left(e^{-2s}\frac{\pi}{s^2+\pi^2}\right) + 5L^{-1}\left(e^{-s}\frac{1}{s^2+\pi^2}\right) \quad (1)$$

$\cancel{=2}$

Date: / /

$$L^{-1}\left(e^{-2s}\frac{\pi}{s^2+\pi^2}\right)$$

$$L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t = f_1(t)$$

By 2nd S.T

$$L^{-1}\left(e^{-2s}\frac{\pi}{s^2+\pi^2}\right) = f_1(t-2)\, u(t-2)$$

$$= \sin(\pi(t-2))\, u(t-2) \quad (2)$$

$$L^{-1}\left(e^{-s}\frac{1}{s^2+\pi^2}\right)$$

$$L^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{\sin \pi t}{\pi} = f_2(t)$$

By 2nd ST

$$L^{-1}\left(e^{-s}\frac{1}{s^2+\pi^2}\right) = f_2(t-1)\, u(t-1)$$

$$= \frac{\sin \pi(t-1)\, u(t-1)}{\pi}$$

$\therefore$ From (1), (2) & (3)

$$\text{Ans} = \sin(\pi(t-2))\, u(t-2) + \frac{5}{\pi}\sin(\pi(t-1))\, u(t-$$

Qn. Find $L^{-1}\left[\frac{e^{-3s}}{(s+2)^2}\right]$

Soln.

Here $f(t) = L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$ By 1^{st} ST

$= e^{-2t} t$

$\therefore L^{-1}\left[e^{-3s} \frac{1}{(s+2)^2}\right] = f(t-3)\, u(t-3)$ By 2^{nd} ST

$= e^{-2(t-3)} (t-3)\, u(t-3)$

Dirac's Delta Function.

Consider the function

$$f_k(t-a) = \begin{cases} 1/k & \text{for } a \le t \le a+ \\ 0 & \text{otherwise} \end{cases} \quad \text{—} (1)$$

This function represents a force of magnitude $\frac{1}{k}$ acting from $t=a$ to $t=a$ where k is +ve and small. In mechanics, the integral of a force acting over a time interval $(a, a+k)$

Date: / / /

is called the impulse of the force

~~Define~~ $I_k = \int_0^{\infty} f_k(t-a)\,dt$

$$= \int_a^{a+k} \frac{1}{k}\,dt = \frac{1}{k}(t)_a^{a+k} = 1 \quad (2)$$

Here the impulse I_k of the force f_k is 1.

Dirac delta function or unit impulse function $\delta(t-a)$ is defined by

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

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From (1) & (2) by taking limit as $k \to 0$ we get

$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases} \quad \& \quad \int_0^\infty \delta(t-a)\,dt = 1$$

Sifting property

For a given function $f(t)$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

proof

$$f(t)\,\delta(t-a) = f(a)\,\delta(t-a),\ \forall t$$

$\therefore \int_0^\infty f(t)\,\delta(t-a)\,dt$	$\because$ if $t \neq a$
	$f(t)\delta(t-a) = 0$ &
$= \int_0^\infty f(a)\,\delta(t-a)\,dt$	$f(a)\,\delta(t-a) = 0$
	if $t = a$
$= f(a)\int_0^\infty \delta(t-a)\,dt$	$f(t)\,\delta(t-a) = f(a)\delta($

$$= f(a)$$

Date: / / /

Find $L[\delta(t-a)]$

$$L[\delta(t-a)] = \int_0^\infty e^{-st}\,\delta(t-a)\,dt$$

$$= e^{-sa} \quad \text{(By sifting prop)}$$

Aliter

$$L[\delta(t-a)] = \lim_{k\to 0} L[f_k(t-a)]$$

$$= \lim_{k\to 0} L\left(\frac{1}{k}\left(u(t-a) - u(t-(a+k))\right)\right)$$

$$= \lim_{k\to 0} \frac{1}{k}\left[L(u(t-a)) - L(u(t-(a+k)))\right]$$

$$= \lim_{k\to 0} \frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right]$$

$$= \frac{e^{-as}}{s} \lim_{k\to 0}\left[\frac{1-e^{-ks}}{k}\right] \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \frac{e^{-as}}{s} \lim_{k\to 0} \frac{se^{-ks}}{1}$$

$$= e^{-as}$$

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Determine the response of the damped mass-spring system under a square wave, modeled by

$$y'' + 3y' + 2y = u(t-1) - u(t-2),$$
$$y(0) = 0, \; y'(0) = 0$$

Soln.

G.T $\quad y'' + 3y' + 2y = u(t-1) - u(t-2)$

$$L(y'') + 3L(y') + 2L(y) = L(u(t-1)) - L(u(t-2))$$

$$s^2 L(y) - s\,y(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$s^2 L(y) + 3s\,L(y) + 2L(y) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$(s^2+3s+2)\,L(y) = \frac{e^{-s} - e^{-2s}}{s}$$

$$\therefore L(y) = \frac{e^{-s} - e^{-2s}}{s(s^2+3s+2)}$$

$$= \frac{e^{-s} - e^{-2s}}{s(s+1)(s+2)}$$

Date: / / /

$$= \left[e^{-s} - e^{-2s}\right] \times F(s) \quad \text{where}$$

$$F(s) = \frac{1}{s(s+1)(s+2)}$$

$$y(t) = L^{-1}\left[e^{-s}F(s) - e^{-2s}F(s)\right]$$

$$= L^{-1}(e^{-s}F(s)) - L^{-1}(e^{-2s}F(s))$$

$$= u(t-1)f(t-1) - u(t-2)f(t-2) \quad \text{— (1)}$$

By 2^{nd} S.T

where $f(t) = L^{-1}[F(s)]$

$$F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

put $s=0$ $\quad 2A = 1 \Rightarrow A = \frac{1}{2}$

$s=-1$ $\quad -B = 1 \Rightarrow B = -1$

$s=-2$ $\quad 2C = 1 \Rightarrow C = \frac{1}{2}$

$$\therefore F(s) = \frac{1/2}{s} - \frac{1}{(s+1)} + \frac{1/2}{(s+2)}$$

$$\therefore L^{-1}[F(s)] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+}\right.$$

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$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$\therefore f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \quad \text{(2)}$$

$\therefore$ From eqn (1)

$$y(t) = \begin{cases} 0 & \text{if } 0<t<1 \\ f(t-1) & \text{if } 1<t<2 \\ f(t-1) - f(t-2) & \text{if } t>2 \end{cases}$$

From (2)

$$f(t-1) = \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

$$= \frac{1}{2} - e^{1-t} + \frac{1}{2} e^{2(1-t)}$$

$$f(t-2) = \frac{1}{2} - e^{2-t} + \frac{1}{2} e^{2(2-t)}$$

$$f(t-1) - f(t-2) = -e^{1-t} + e^{2-t} + \frac{1}{2} e^{2(1-t)} - \frac{1}{2} e^{2(2-t)}$$

$$\therefore y(t) = \begin{cases} 0, & 0<t<1 \\ \frac{1}{2} - e^{1-t} + \frac{1}{2} e^{2(1-t)}, & 1<t<2 \\ -e^{1-t} + e^{2-t} + \frac{1}{2} e^{2(1-t)} - \frac{1}{2} e^{2(2-t)}, & t>2 \end{cases}$$

Date: / / /

Solve the IVP $y'' + 3y' + 2y = \delta(t-1)$,
$y(0) = 0, \; y'(0) = 0$

Soln :

G.T $\quad y'' + 3y' + 2y = \delta(t-1)$

$$L(y'' + 3y' + 2y) = L[\delta(t-1)]$$

$$\Rightarrow (s^2 + 3s + 2)\, L(y) = e^{-s}$$

$$\Rightarrow \quad L(y) = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s} F(s) \quad \text{—(1)}$$

where $F(s) = \dfrac{1}{(s+1)(s+2)}$

$$F(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

put $s = 1$, $\quad 1 = A$

$s = -2$, $\quad -1 = B$

$$\therefore \; F(s) = \frac{1}{(s+1)} - \frac{1}{(s+2)} \quad \text{—(2)}$$

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From ①

$$y(t) = L^{-1}\left[e^{-s}F(s)\right]$$

$$= u(t-1)\,f(t-1) \quad \text{By } 2^{nd} \text{ ST}$$

where $f(t) = L^{-1}[F(s)]$

From ② $\quad f(t) = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$

$$= e^{-t} - e^{-2t}$$

$$\therefore y(t) = \begin{cases} 0 & 0<t<1 \\ f(t-1) & t>1 \end{cases}$$

$$= \begin{cases} 0, & 0<t<1 \\ e^{1-t} - e^{2(1-t)}, & t>1 \end{cases}$$

Date : / / /

Qn Solve $y'' + 5y' + 6y = u(t-1)$, $y(0)=0$, $y'(0)=1$

Qn Solve $y'' + 16y = 4\delta(t-3\pi)$, $y(0)=0$, $y'(0)=0$

Date / /

★ Convolution

The convolution of two functions $f(t)$ and $g(t)$ is defined by

$$(f*g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t g(u)\, f(t-u)\, du$$

$$= (g*f)(t)$$

$$\Big[(f*g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

$v = t-u$, $dv = -du$

$$= \int_t^0 f(t-v)\, g(v)\, (-dv)$$

$$= -\int_t^0 g(v)\, f(t-v)\, dv$$

$$= \int_0^t g(v)\, f(t-v)\, dv$$

$$= (g*f)(t) \Big]$$

Remark: This is called the commutative property of

Date: / / /

convolution.

Qn Find the convolution of e^t and t

Soln :

Let $f(t) = e^t$ & $g(t) = t$

$$\therefore (f*g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^u (t-u)\, du$$

$$= t\int_0^t e^u du - \int_0^t u e^u du$$

$$= t\left[e^u\right]_0^t - \left\{ u e^u - 1 e^u \right\}_0^t$$

$$= t(e^t - 1) - ((te^t - 0) - (e^t - 1))$$

$$= \cancel{te^t} - t - \cancel{te^t} + e^t - 1$$

$$= e^t - t - 1$$

Qn Find the convolution of t and 1

Soln.

Let $f(t) = t$, $g(t) = 1$

$$\therefore (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

$$t * 1 = \int_0^t u \cdot 1\, du$$

$$= \left[\frac{u^2}{2}\right]_0^t$$

$$= \frac{t^2}{2}$$

Remark: In general $(f * 1)(t) \neq f(t)$

Remark Also $\mathcal{L}[f(t)g(t)] \neq \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$

Eg: Consider $f(t) = e^t$ & $g(t) = 1$

$$f(t)g(t) = e^t$$

$$\mathcal{L}[f(t)g(t)] = \mathcal{L}(e^t) = \frac{1}{s-1}$$

$$\mathcal{L}[f(t)] = \frac{1}{s-1}, \quad \mathcal{L}[g(t)] = \frac{1}{s}$$

$$\therefore \mathcal{L}[f(t)]\,\mathcal{L}[g(t)] = \frac{1}{(s-1)s}$$

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Date: / / /

* Convolution Theorem

$$L[(f*g)(t)] = L[f(t)]\,L[g(t)] = F(s)G(s)$$

Thus $$L^{-1}[F(s)G(s)] = (f*g)(t),$$

where $F(s) = L(f(t))$
$G(s) = L(g(t))$

Qn Use convolution theorem to find $L^{-1}\left[\dfrac{1}{s(s-a)}\right]$

Soln

Let $F(s) = \dfrac{1}{s}$ $\quad G(s) = \dfrac{1}{s-a}$

$\therefore$ By convolution theorem we have

$$L^{-1}[F(s)\,G(s)] = (f*g)(t) \quad \text{where} \quad (1)$$

$f(t) = L^{-1}[F(s)]$
$g(t) = L^{-1}[G(s)]$

$$\therefore \quad f(t) = L^{-1}\left(\frac{1}{s}\right) = 1$$

$$g(t) = L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$(f*g)(t) = (g*f)(t)$$

$$= \int_0^t g(u)\, f(t-u)\, du$$

$$= \int_0^t e^{au} \times 1\, du$$

$$= \left[\frac{e^{au}}{a}\right]_0^t$$

$$= \frac{e^{at}-1}{a}$$

Substituting in eqn ①

$$L^{-1}\left[\frac{1}{s(s-a)}\right] = \frac{e^{at}-1}{a}$$

Date: / / /

Q. Evaluate $L^{-1}\left[\frac{1}{(s^2+9)^2}\right]$

Soln

$$L^{-1}\left[\frac{1}{(s^2+9)^2}\right] = L^{-1}\left[\frac{1}{(s^2+3^2)} \times \frac{1}{(s^2+3^2)}\right]$$

$$= (f*g)(t)$$

Let $F(s) = \frac{1}{s^2+3^2}$, $G(s) = \frac{1}{s^2+3^2}$

where $f(t) = L^{-1}[F(s)]$
$g(t) = L^{-1}[G(s)]$

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{\sin 3t}{3}$$

|||ly $g(t) = \frac{\sin 3t}{3}$

$$\therefore L^{-1}\left[\frac{1}{(s^2+9)^2}\right] = (f*g)(t)$$

$$= \int_0^t \frac{\sin 3u}{3} \frac{\sin 3(t-u)}{3} du$$

$$= \frac{1}{9}\int_0^t \sin 3u \sin(3t-3u)\, du$$

$$\left[\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]\right]$$

Date: / /

$$= \frac{1}{9}\int_0^t \frac{1}{2}\left[\cos(6u-3t) - \cos 3t\right]du$$

$$= \frac{1}{18}\left[\frac{\sin(6u-3t)}{6} - u\cos 3t\right]_0^t$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{6} - t\cos 3t - \left(\frac{-\sin 3t}{6} - 0\right)\right]$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

Using convolution theorem

Qn Find $L^{-1}\left[\frac{a}{s^2(s^2+a^2)}\right]$

Qn Find $L^{-1}\left[\frac{s}{(s+3)^3}\right]$

Qn Find $L^{-1}\left[\frac{1}{(s^2+2s+5)^2}\right]$

Qn Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

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